ON THE THEORY OF SHALLOW SHELLS

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1. On the differential equations of shallow shells in the case of normal loading. The differential equations of the technical theory of shallow shells can, in the case of loading Z normal to the surface of the shell, be written in the following form as suggested by Vlasov [1],[2]

$$\Delta\Delta\varphi + Eh\Delta_k w = 0, \qquad D\Delta\Delta w - \Delta_k\varphi = Z, \qquad D = \frac{Eh^3}{12(1-\nu^2)} \tag{1.1}$$

The unknowns here are the normal deflection w and the stress function ϕ, while the loading components X and Y are taken to be zero; the shell thickness h is considered to be constant, E is the modulus of elasticity of the material and ν is Poisson's ratio. The operators Δ and Δ_k are defined by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \Delta_k = k_x\frac{\partial^2}{\partial y^2} + k_y\frac{\partial^2}{\partial x^2} - 2k_{xy}\frac{\partial^2}{\partial x\,\partial y} \tag{1.2}$$

Here k_x, k_y represent the bending curvature and k_{xy} is the torsional curvature of the shell surface.

Denoting by F the function determining the surface of the shallow shell, we have approximately

$$k_x = \frac{\partial^2 F}{\partial x^2}, \qquad k_y = \frac{\partial^2 F}{\partial y^2}, \qquad k_{xy} = \frac{\partial^2 F}{\partial x\,\partial y} \tag{1.3}$$

$$\frac{\partial k_x}{\partial y} - \frac{\partial k_{xy}}{\partial x} = 0, \qquad \frac{\partial k_y}{\partial x} - \frac{\partial k_{xy}}{\partial y} = 0$$
$$\frac{\partial^2 k_x}{\partial y^2} + \frac{\partial^2 k_y}{\partial x^2} - 2\frac{\partial^2 k_{xy}}{\partial x\,\partial y} = 0 \tag{1.4}$$

The normal stress resultants N_x and N_y and the tangential stress resultant S, the bending stress moment resultants M_x and M_y and the torsional stress moment resultant M_{xy} are defined by

$$N_x = \frac{\partial^2 \varphi}{\partial y^2}, \qquad M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$N_y = \frac{\partial^2 \varphi}{\partial x^2}, \qquad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \tag{1.5}$$

$$S = -\frac{\partial^2 \varphi}{\partial x \, \partial y}, \qquad M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \, \partial y}$$

The surface of the shell, the system of coordinates and the positive directions of the stress resultants are shown in Fig. 1. The plane Oxy is the coordinate plane of the shallow shell. The moving system of coordinates $O'x'y'z'$ is connected with an arbitrary point of the shell in such a manner that the axes $x'y'$ are situated in the tangential plane, while z' is directed along the normal to the shell surface. The axes x' and y' are located respectively in the planes y = const and x = const. The positive directions of the displacements u, v, w and of the components X, Y, Z of the distributed surface loading respectively coincide with the directions of the coordinate axes x', y', z'.

Introducing the scalar function W by means of the formulas

$$w = \triangle\triangle W, \qquad \varphi = -Eh\triangle_k W \tag{1.6}$$

Vlasov [1] transforms the system (1.1) into

$$D\triangle\triangle\triangle\triangle W + Eh\triangle_k\triangle_k W = Z \tag{1.7}$$

It is convenient to use this equation in cases when the curvatures k_x and k_y are constant, while the curvature k_{xy} is zero.

Attention must be paid to the fact that the formulas (1.6) and (1.7) become incorrect in the case of a spherical shell. Indeed, if $k_x = k_y = 1/R$ = const and $k_{xy} = 0$, equations (1.6) assume the form

$$w = \triangle\triangle W, \qquad \varphi = -\frac{Eh}{R}\triangle W, \qquad \triangle_k = \frac{1}{R}\frac{\partial^2}{\partial y^2} + \frac{1}{R}\frac{\partial^2}{\partial x^2} = \frac{1}{R}\triangle \tag{1.8}$$

Eliminating W from these equations, we get

$$w = -\frac{R}{Eh}\triangle\varphi \tag{1.9}$$

On the other hand, the first equation of the original system (1.1) gives the formula

$$\triangle w = -\frac{R}{Eh}\triangle\triangle\varphi \tag{1.10}$$

It is obvious that the formulas (1.9) and (1.10) are not identical, as they should be. Consider the following example: if $\Delta\phi = 0$, then we have $w = 0$ according to (1.9) and $\Delta w = 0$ according to (1.10). Thus, in the special case of the spherical shell the equations

$$D\triangle\triangle\triangle\triangle w + \frac{Eh}{R^2}\triangle\triangle W = Z \tag{1.11}$$

$$D\triangle\triangle w + \frac{Eh}{R^2} w = Z \tag{1.12}$$

obtained from (1.7) and (1.8) are not correct, and thus can lead to erroneous results.

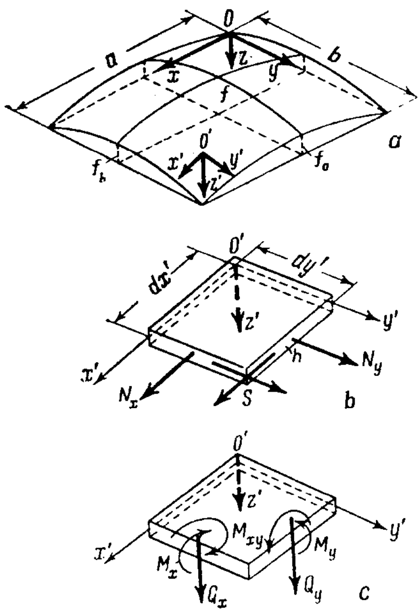

Fig. 1.

In the case of a spherical shell we have to use the original system of differential equations (1.1), or a new function W^* defined by the formulas

$$\triangle w = \triangle W^*, \qquad \triangle\varphi = -\frac{Eh}{R} W^* \tag{1.13}$$

Eliminating W^* from the two formulas given above we obtain (1.10), as it should be.

Substituting the formulas (1.13) into the second equation of the system (1.1) we find

$$D\triangle\triangle W^* + \frac{Eh}{R^2} W^* = Z \tag{1.14}$$

The equations (1.13), unlike (1.8) and (1.12), are correct, and there is no danger of their leading to faulty results.

2. On the differential equations of shallow shells in the case of arbitrary loading. In the case of arbitrary loading, a system of three differential equations, also suggested by Vlasov [2] can be used; the unknowns occurring in these equations are the components u, v, w of the complete displacement. The system just indicated is, however, relatively involved and inconvenient for practical application in most cases. It is, therefore, desirable to derive a generalization of the system (1.1) applicable to any loading.

For the special case of circularly cylindrical shells such a generalization is given in the book by Ruediger and Urban [4].

It so happens that the desired generalization can also be easily obtained for the general case of a shallow shell with arbitrary curvatures. It is only necessary to replace the first two of the relationships (1.5) by the following:

$$N_x = \frac{\partial^2 \varphi}{\partial y^2} - \int X dx, \qquad N_y = \frac{\partial^2 \varphi}{\partial x^2} - \int Y dy \tag{2.1}$$

Now we can proceed as indicated by Vlasov [2] and confine ourselves solely to linear terms. In this manner we find

$$\begin{gathered} \triangle\triangle\varphi + Eh\Delta_k w = \int \frac{\partial^2 X}{\partial y^2}\, dx + \int \frac{\partial^2 Y}{\partial x^2}\, dy - \nu\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \\ D\triangle\triangle w - \Delta_k \varphi = Z - k_x \int X dx - k_y \int Y dy \end{gathered} \tag{2.2}$$

Instead of the system (2.2) we may use the differential equation with complex coefficients obtained by substitution of

$$u^* = w + i\alpha\varphi \qquad \left(\alpha = \frac{\sqrt{12(1-\nu^2)}}{Eh^2},\ i = \sqrt{-1}\right) \tag{2.3}$$

This leads to

$$\begin{aligned} \triangle\triangle u^* + i\frac{\sqrt{12(1-\nu^2)}}{h} \Delta_k u^* = \frac{1}{D}\Big\{Z - k_x \int X dx - k_y \int Y dy + \\ + i\frac{h}{\sqrt{12(1-\nu^2)}}\left[\int \frac{\partial^2 X}{\partial y^2}\, dx + \int \frac{\partial^2 Y}{\partial x^2}\, dy - \nu\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\right]\Big\} \end{aligned} \tag{2.4}$$

After having carried out the integration and determined u^*, we find the deflection w and the stress function ϕ by separating the real and imaginary parts of u^*.

If the stress resultants are known, the displacement components of the shell are determined by the following system of differential equations:

$$\begin{aligned} \frac{\partial u}{\partial x} - k_x w &= \frac{1}{Eh}(N_x - \nu N_y) \\ \frac{\partial v}{\partial y} - k_y w &= \frac{1}{Eh}(N_y - \nu N_x) \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2k_{xy} w &= \frac{2(1+\nu)}{Eh} S \end{aligned} \tag{2.5}$$

3. The case of uniformly distributed tangential loading. If the plane view of the shell is rectangular and if the torsional curvature is zero, then the solution of the problem is very simple. We assume the usual boundary conditions, namely: the shell is loosely supported, while the stringers are rigid in their own plane and flexible outside it.

Without loss of generality we assume that the tangential loading is acting parallel to one coordinate axis. In this case we may write

$$X = X_0 = \text{const}, \qquad Y = Z = 0, \qquad k_{xy} = 0, \qquad \int X_0 dx = X_0 x$$

$$w,\ v,\ M_x,\ N_x = 0 \text{ when } x = 0 \text{ and } x = a; \qquad w,\ u,\ M_y,\ N_y = 0 \text{ when } y = 0 \text{ and } y = b$$

The system (2.2) assumes the form

$$\triangle\triangle\varphi + Eh\triangle_k w = 0, \qquad D\triangle\triangle w - \triangle_k\varphi = -k_x X_0 x \tag{3.1}$$

It is easy to verify that this system and the boundary conditions for w, M_x, M_y, N_x and N_y are satisfied if

$$w = 0, \qquad \varphi = \frac{1}{2} X_0 x y^2 + Cxy \tag{3.2}$$

The constant C is to be determined from the boundary conditions for u and v. All stress resultants, except the tangential stress resultant S are zero. We find

$$S = -\frac{\partial^2\varphi}{\partial x \partial y} = -X_0 y - C \tag{3.3}$$

Substituting the values of w and of the stress resultants into the system (2.5) we get

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{2(1+\nu)}{Eh}(X_0 y + C) \tag{3.4}$$

From the first two equations of this system we see that u can be a function of y only, and v of x only. In this case the third equation of the system (3.4) becomes

$$\frac{du}{dy} + \frac{dv}{dx} = -\frac{2(1+\nu)}{Eh}(X_0 y + C) \tag{3.5}$$

For the displacements u and v we must assume

$$u = -\frac{1+\nu}{Eh} X_0 y^2 + C_1 y + C_2, \qquad v = C_3 x + C_4 \tag{3.6}$$

where C_1, C_2, C_3, C_4 are constants of integration. Substituting (3.6) into (3.5) we find

$$C_1 + C_3 = -\frac{2(1+\nu)}{Eh} C \tag{3.7}$$

On the basis of the boundary conditions assumed for u and v we get

$$C_1 = \frac{(1+\nu)b}{Eh} X_0, \qquad C_2 = C_3 = C_4 = 0 \tag{3.8}$$

Taking into account (3.2), (3.3), (3.6), (3.7), (3.8), we obtain

$$u = \frac{1+\nu}{Eh} y(b-y) X_0, \qquad \varphi = -\frac{1}{2}(b-y) xy X_0, \qquad S = \frac{1}{2}(b-2y) X_0 \tag{3.9}$$

All other displacements and stress resultants are zero. Note that the formulas obtained are independent of the curvatures k_x and k_y, which thus remain arbitrary. This is a case in which the shallow shell behaves like a deep beam.

The problem becomes more difficult if the torsional curvature k_{xy} is not zero. In this case it is convenient to assume

$$w = w_0 + w_1, \qquad \varphi = \varphi_0 + \varphi_1 \tag{3.10}$$

Here we assume for w_0 and ϕ_0 the results (3.9), i.e.

$$w_0 = 0, \qquad \varphi_0 = -\frac{1}{2}(b-y) xy X_0 \tag{3.11}$$

w_0 and ϕ_0 satisfy the assumed boundary conditions exactly. Including (3.10) into the system (3.1), we get

$$\triangle\triangle\varphi_1 + Eh\triangle_k w_1 = 0, \qquad D\triangle\triangle w_1 - \triangle_k \varphi_1 = k_{xy}(b-2y) X_0 \tag{3.12}$$

by virtue of which the further treatment of the problem reduces to an investigation of the structure under the action of the fictitious normal loading

$$Z^* = k_{xy}(b-2y) X_0$$

4. Shell of constant curvature under arbitrary tangential loading. This problem has been studied by Oniashvili [3] under the assumption of vanishing torsional curvature in connection with the usual boundary conditions (loosely supported shell with stringers of the kind stated above). He derived his formulas immediately from the conditions of equilibrium. The discussion of the same problem given below is based upon the relations

(2.1) and (2.2). The solution is obtained with the aid of double trigonometric series. To simplify the presentation we will study the influence of one arbitrary term of the series only, for the loading X.

On this basis we may write

$$X = X_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \qquad \begin{pmatrix} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \end{pmatrix} \tag{4.1}$$

We consider the constants X_{mn} to be known quantities, while for w and ϕ we assume

$$w = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \qquad \varphi = B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{4.2}$$

The expressions (4.1) and (4.2) satisfy the assumed boundary conditions precisely.

Substituting (4.1) and (4.2) into the system (2.2), we obtain a system of linear algebraic equations, from which

$$\begin{aligned} A_{mn} &= -\frac{a^5 m}{D\pi^5} \frac{m^2 (k_1 + \nu k_2) + \lambda^2 n^2 [(2+\nu) k_1 - k_2]}{(m^2 + \lambda^2 n^2)^4 + \mu (k_2 m^2 + k_1 \lambda^2 n^2)^2} X_{mn} \\ B_{mn} &= -\frac{a^3}{\pi^3 m} \frac{(m^2 + \lambda^2 n^2)(\lambda^2 n^2 - \nu m^2) + \mu k_1 (k_2 m^2 + k_1 \lambda^2 n^2)}{(m^2 + \lambda^2 n^2)^4 + \mu (k_2 m^2 + k_1 \lambda^2 n^2)^2} X_{mn} \end{aligned} \tag{4.3}$$

where

$$\lambda = \frac{a}{b}, \qquad \mu = \frac{12 (1 - \nu^2) a^4}{\pi^4 h^2}, \qquad k_x = k_1 = \text{const}, \qquad k_y = k_2 = \text{const}$$

Using the formulas (1.5), (2.1), (2.5) for the displacements and the stress resultants, we get

$$\begin{aligned} u = \frac{a^2}{\pi^2 Eh} \frac{1}{\vartheta_1 (m, n)} \{\vartheta_2 (m, n) [(1 - \nu^2) m^2 + 2 (1 + \nu) \lambda^2 n^2] + \\ + \mu [(k_1^2 + k_2^2 + 2\nu k_1 k_2) m^2 + 2 (1 + \nu) k_1^2 \lambda^2 n^2]\} X_{mn} \cos \frac{m\pi x}{a} \sin \frac{\pi x y}{b} \\ v = -\frac{a^2 m \lambda n}{\pi^2 Eh} \frac{(1 + \nu)^2 \vartheta_2 (m, n) + \mu (k_1 - k_2)^2}{\vartheta_1 (m, n)} X_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \end{aligned} \tag{4.4}$$

$$\begin{aligned} N_x &= -\frac{am}{\pi} \frac{\vartheta_2 (m, n) [m^2 + (2 + \nu) \lambda^2 n^2] + k_2 \vartheta_3 (m, n)}{\vartheta_1 (m, n)} X_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ N_y &= \frac{am}{\pi} \frac{\vartheta_2 (m, n) \vartheta_4 (m, n) + k_1 \vartheta_3 (m, n)}{\vartheta_1 (m, n)} X_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ S &= \frac{a\lambda n}{\pi} \frac{\vartheta_2 (m, n) \vartheta_4 (m, n) + k_1 \vartheta_3 (m, n)}{\vartheta_1 (m, n)} X_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \end{aligned} \tag{4.5}$$

$$\begin{aligned} M_x &= -\frac{a^3 m (m^2 + \nu \lambda^2 n^2)}{\pi^3} \frac{\vartheta_5 (m, n)}{\vartheta_1 (m, n)} X_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ M_y &= -\frac{a^3 m (\lambda^2 n^2 + \nu m^2)}{\pi^3} \frac{\vartheta_5 (m, n)}{\vartheta_1 (m, n)} X_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ M_{xy} &= -\frac{(1 - \nu) a^3 m^2 \lambda n}{\pi^3} \frac{\vartheta_5 (m, n)}{\vartheta_1 (m, n)} X_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \end{aligned} \tag{4.6}$$

where

$$\vartheta_1(m, n) = (m^2 + \lambda^2 n^2)^4 + \mu (k_2 m^2 + k_1 \lambda^2 n^2)^2$$

$$\vartheta_2(m, n) = (m^2 + \lambda^2 n^2)^2, \qquad \vartheta_3(m, n) = \mu (k_2 m^2 + k_1 \lambda^2 n^2)$$

$$\vartheta_4(m, n) = \lambda^2 n^2 - \nu m^2, \qquad \vartheta_5(m, n) = m^2 (k_1 + \nu k_2) + \lambda^2 n^2 [(2 + \nu) k_1 - k_2]$$

Summation with respect to m and n permits the influence of all terms of the series for the tangential loading X to be estimated.

It should be noted that, if X depends on y only ($m = 0$), the displacements w and v and the resultants N_x and N_y are zero everywhere on the shell surface. Furthermore, if $X = X_0 = \text{const}$ and

$$X = \sum_n X_{0n} \sin \frac{n\pi y}{b} = \frac{4X_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi y}{b}$$

then by improving the convergence of the infinite trigonometric series by the method of Krylov, we obtain formula (3.9), as was to be expected.

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